§3.

3.1 $\mathbb{P}^2 \supset \mathbb{A}^2 = \mathbb{P}^2 \setminus l_\infty$

$M(r,n)$ = framed moduli space of torsion free sheaves (E,φ) with $c_2(E)=n$

$$\varphi : E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}$$

Comment : $\mathbb{A}^2$ = open K3 . Study of $M(r,n)$ was motivated by Mukai's work on V.b's on K3

Th $M(r,n)$ is a nonsingular quasiprojective variety of $\dim = 2rn$

1^{st} proof :
- develop the theory of stable pairs
- deformation theory is controlled by $\mathrm{Ext}^i(E, E(-l_\infty))$
- existence of the framing

$\Longrightarrow$ stability

$$\mathrm{Ext}^0(E, E(-l_\infty)) = \mathrm{Ext}^2(E, E(-l_\infty)) = 0$$
$$\chi(\mathrm{Ext}^\bullet(E, E(-l_\infty))) = -2rn$$

2^{nd} proof :

quiver description :

V : a cpx vector space of $\dim = n$

W : 〃 r

$B_1, B_2 \in \mathrm{End}(V)$

$a : W \to V$, $b : V \to W$

$$\begin{tikzcd} V \arrow[loop left, "B_2"] \arrow[loop right, "B_1"] \arrow[d, shift right, "b"'] \\ W \arrow[u, shift right, "a"'] \end{tikzcd}$$

$G = GL(V) \curvearrowright \mathrm{End}(V)^{\oplus 2} \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W)$

↖ symplectic vector sp

- moment map $\mu(B_1, B_2, a, b) = [B_1, B_2] + ab$

- (B_1, B_2, a, b) is stable

$\Longleftrightarrow$ ⌜ $\forall S \subset V$ subsp s.t. $\mathrm{Im}\, i \subset S$, $B_\alpha(S) \subset S$ must be $S = V$ ⌟

Th. $M(r,n) = \mu^{-1}(0)^{\text{stable}} / G$
(sketch of the proof)
Consider the following cpx on $\mathbb{P}^2$

$$V \otimes \mathcal{O}(-1) \xrightarrow{\alpha} V^{\oplus 2} \oplus W \otimes \mathcal{O} \xrightarrow{\beta} V \otimes \mathcal{O}(1)$$

$$\alpha \mapsto \begin{bmatrix} B_1 z - x \\ B_2 z - y \\ b z \end{bmatrix} \qquad \beta \mapsto \begin{bmatrix} -(B_2 z - y) & (B_1 z - x) & a z \end{bmatrix}$$

- $\mu = 0 \Rightarrow \beta\alpha = 0$
- stability $\Rightarrow \beta$: surjective $(\forall [x:y:z])$
- α : injective $\Leftarrow$ injective at $z = 0$
  (but not a subbundle)

$\Longrightarrow$ Define $E = \operatorname{Ker}\beta / \operatorname{Im}\alpha$
$\alpha, \beta|_{z=0}$ gives a framing.

Inverse transform : $V = H^1(E(-l_\infty))$ etc
$W =$ fiber at l_∞

(2^{nd} proof of the smoothness)
<u>Claim</u> ① $d\mu$ is surjective on a stable point
② $G \curvearrowright$ {stable points} freely
∵ ① $d\mu(\delta B_1, \delta B_2, \delta a, \delta b) = [\delta B_1, B_2] + [B_1, \delta B_2] + \delta a \cdot b + a \delta b$
∴ $\xi \perp \operatorname{Im} d\mu \Leftrightarrow [B_\alpha, \xi] = 0$
$b\xi = 0, \ \xi a = 0$
∴ $\operatorname{Ker}\xi \supset \operatorname{Im} a$ & inv. under B_α
∴ stability $\Rightarrow \xi = 0$

② Stab $\ni g$ Consider $\xi = \mathrm{id} - g$.
$\Rightarrow \xi = 0$ //

Cor of this construction :

$\exists\, \pi$: projective morphism

$$M(r,n) \longrightarrow M_0(r,n) = \mu^{-1}(0) /\!/ G$$
$$= \mathrm{Spec}(\mathbb{C}[\mu^{-1}(0)]^G)$$

At the level of sets,

$M_0(r,n) = \{$closed G-orbits$\}$

= semisimple representations of the quiver with the rel. $\mu = 0$

$\wr$

direct sum of simple representations

It is not difficult to classify all semisimple rep.

<u>Prop.</u> simple $\Leftrightarrow$ either of the following

a) (B_1, B_2, a, b) stable & costable

b) $W = 0$, $\dim V = 1$ & $B_1 = x$, $B_2 = y$

loc. free framed sheaf

$\wr$

point in $\mathbb{A}^2$

$$\therefore \quad M_0(r,n) = \coprod_{0 \le k \le n} M_0^{l.f.}(r,k) \times S^{n-k}\mathbb{A}^2$$

symmetric prod.

This space is called the Uhlenbeck space.

<u>Example</u> $r = 1$

$$\begin{aligned} M(r,n) &= \mathrm{Hilb}^n \mathbb{C}^2 \\ &\downarrow \\ M_0(r,n) &= S^n \mathbb{C}^2 \end{aligned}$$

3.2

Let $\mathbb{T} = T \times T^2 \curvearrowright M(r,n), M_0(r,n)$

$\uparrow$ change of framing $\qquad$ action on the base

$$H^*_{\mathbb{T}}(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2, \vec{a}] \qquad \vec{a} = (a_1, \cdots, a_r)$$

<u>Claim</u> $M_0(r,n)^{\mathbb{T}} = \{0\}$

$i_0 : \{0\} \hookrightarrow M_0(r,n)$

Define the instanton partition function (Nekrasov) by

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum \Lambda^{4nr}\, i_{0*}^{-1} [M_0(r,n)]$$

$$\in \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{a})[[\Lambda]]$$

Consider the following diagram :

$$\begin{tikzcd} M(r,n) \arrow[d, "\pi"'] & M(r,n)^{\mathbb{T}} \arrow[l, hook', "i"'] \arrow[d, "\pi^{\mathbb{T}}"] \\ M_0(r,n) \arrow[d, "a"'] & \{0\} \arrow[l, hook', "i_0"] \arrow[dl, equal] \\ pt & \end{tikzcd}$$

$M(r,n)^{\mathbb{T}} = \{\vec{Y} = (Y_1, \cdots, Y_r) \mid \sum |Y_\alpha| = n\}$

will be shown soon $\qquad$ Young diagram

$$a \circ i_0 = id \quad \therefore \quad i_{0*}^{-1} = \text{“} a_* = \int_{M_0(r,n)} \text{”}$$

not defined as a is not proper.

<u>Lemma</u> (1) $\pi_* i_* = i_{0*} \pi^{\mathbb{T}}_*$

(2) $\pi_*[M(r,n)] = [M_0(r,n)]$

Th. $i_{0*}^{-1}[M_0(r,n)] = \pi_*^{T} i_*^{-1}[M(r,n)]$

$$= \sum_{\vec{Y}} \frac{1}{e(T_{\vec{Y}} M(r,n))}$$

This was the original definition

Example $r=1$

$$M(r,n) = \mathrm{Hilb}^n \mathbb{C}^2$$
$$\downarrow$$
$$M_0(r,n) = S^n \mathbb{C}^2$$

$$i_{0*}^{-1}[S^n\mathbb{C}^2] = \frac{1}{n!}\tilde{i}_{0*}^{-1}[\mathbb{C}^{2n}] \qquad \mathbb{C}^{2n} \xleftarrow{\tilde{i}_0} \{0\}$$

$$\circlearrowleft \; T^2$$

$$= \frac{1}{n!}\left(\tilde{i}_{0*}^{-1}[\mathbb{C}^2]\right)^n$$

$$= \frac{1}{n!}\left(\frac{1}{\varepsilon_1\varepsilon_2}\right)^n \qquad \varepsilon_1\varepsilon_2 = e(T_0\mathbb{C}^2)$$

Rem $\displaystyle \frac{1}{n!}\frac{1}{(\varepsilon_1\varepsilon_2)^n} = \sum_{Y} \frac{1}{e(T_Y \mathrm{Hilb}^n\mathbb{C}^2)}$

is a nontrivial combinatorial identity.

(Cauchy formula for Jack polynomials)

Nekrasov conjecture

$\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1,\varepsilon_2,\vec{a})\Big|_{\varepsilon_1=\varepsilon_2=0}$ can be computed

by periods of certain explicit hyperelliptic curves.

proved by N+Yoshioka

Nekrasov - Okounkov

Braverman - Etingof

Fixed pts

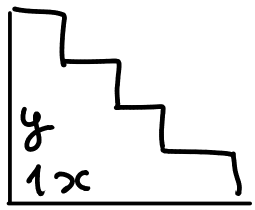
$$M(r,n)^T = \coprod_{n_1+\cdots+n_r=n} M(1,n_1)\times\cdots\times M(1,n_r)$$

(E,φ) is fixed $\iff$ $t\in T$ isom. at $E|_{\ell_\infty}$ extends to E

$\Longrightarrow$ E decomposes into $(I_1,\varphi_1)\oplus\cdots\oplus(I_r,\varphi_r)$

Further I_α is fixed by T^2

$\iff$ $I_\alpha \subset \mathbb{C}[x,y]$ is a monomial ideal



3.3

Here I treat only the rank 2 case, and give a proof following Göttsche - N - Yoshioka. (The sprit of the proof remains the same.)

I assume that we already know that
$\varepsilon_1\varepsilon_2 \log Z$ is regular at $\varepsilon_1 = \varepsilon_2 = 0$.
(This can be proved using the idea below.)

Write $\varepsilon_1\varepsilon_2 \log Z = F_0(a, \Lambda) + \text{higher}$
$a = a_2$, may assume $a_1 + a_2 = 0$

<u>Key definitions</u>

Let $\tau := -\frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial^2 F_0}{\partial a^2} + 8 \log \frac{-2\sqrt{-1}a}{\Lambda}\right)$ $\qquad f := e^{\pi i \tau}$

$u := -\frac{1}{4}\frac{\partial F_0}{\partial \log \Lambda} + a^2$

$\omega := -2\pi\sqrt{-1}\left(\frac{\partial u}{\partial a}\right)^{-1}$, $\omega' = \tau\omega$

Consider a (formal) elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z}\omega \oplus \mathbb{Z}\omega'$ and the associated elliptic function e.g. $\wp$-fct

<u>Rem.</u> $Z = \sum \Lambda^{4n} \int_{M_0(2,n)} 1$

$$\frac{\partial}{\partial \log \Lambda} \log Z = \frac{1}{Z} \sum \underset{\underset{\dim M_0(2,n) \;=\; 4c_2(\mathcal{E})/[\mathbb{P}^2]}{\parallel}}{4n} \Lambda^{4n} \int_{M_0(2,n)} 1$$

later we consider more general integral.

Th. E_C is given by (Seiberg-Witten curve)

$$y^2 = 4x^3 - \left(\frac{4}{3}u^2 - 4\Lambda^4\right)x - \left(\frac{4}{3}u\Lambda^4 - \frac{8}{27}u^3\right)$$

this is define ova $\mathbb{C}[u,\Lambda]$

This determines F_0 :

Note
$$\omega = \omega(u) = \int_A \frac{dx}{y} .$$
$$\omega = -2\pi\sqrt{-1}\left(\frac{\partial u}{\partial a}\right)^{-1}$$

Therefore $a = a(u)$

Then we can write $u = u(a) \Rightarrow F_0$

* Comparison with one given in the literature

$$y^2 = 4x^3 + 4ux^2 + 4x\Lambda^4$$
$$= 4\left(x + \frac{u}{3}\right)^3 + \left(4\Lambda^4 - \frac{4}{3}u^2\right)x - \frac{4}{27}u^3$$
$$= 4\left(x + \frac{u}{3}\right)^3 + \left(4\Lambda^4 - \frac{4}{3}u^2\right)\left(x + \frac{u}{3}\right) - \frac{4}{3}u\Lambda^4 + \frac{8}{27}u^3$$

(sketch of the proof)

$\hat{\mathbb{P}}^2$ = blowup of $\mathbb{P}^2$ at $0 \in \mathbb{A}^2$

Consider $\hat{M}(2,k,n) = \left\{ (E,\varphi) : \text{framed sheaf on } \hat{\mathbb{P}}^2,\ c_1(E) = kC,\ c_2(E) - \frac{c_1(E)^2}{4} = n \right\} / \text{iso.}$

smooth of dim $= 4n$ $\quad T \curvearrowright \hat{M}(2,k,n)$

$\mathcal{E}$: universal sheaf over $\hat{\mathbb{P}}^2 \times \hat{M}(2,k,n)$

$$\mu(C) := c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2 / [C] \in H^2_T(\hat{M}(2,k,n))$$

Idea : Compute in two ways and study ① = ②

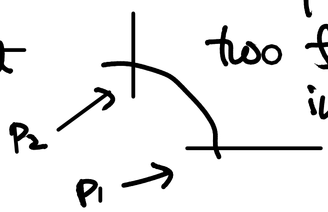
$$\hat{Z}_k(\varepsilon_1,\varepsilon_2,a,t;\Lambda) = \sum_n \Lambda^{4nr} \int_{\hat{M}(2,k,n)} e^{t\mu(C)}$$

① localization

fixed pts $= \{ (\vec{k}, \vec{Y}^1, \vec{Y}^2) \mid \text{constraint} \}$

$$E \cong I_1(k_1 C) \oplus I_2(k_2 C) \qquad k_1 + k_2 = k$$

"monomial ideals at T^2-two fixed pts in $\hat{\mathbb{A}}^2$



$$e(T_{(\vec{k}, \vec{Y}^1, \vec{Y}^2)} \hat{M}) = \text{product of three factors}$$

1) $\mathrm{Ext}^1(\mathcal{O}(k_1 C), \mathcal{O}(k_2 C - \ell_\infty))$
2) contribution from p_1
3) 〃 p_2

1) & $e^{t\mu(C)}$ can be written explicitly

2), 3) can be written by $e(T_{\vec{Y}} M)$ as

locally $(\mathbb{A}^2, 0) \cong (\hat{\mathbb{A}}^2, p_1 \text{ or } p_2)$

T^2-action is modified so that this is T^2-equiv

$\Rightarrow$ $\hat{Z}_k$ can be written in terms of Z as

$$\hat{Z}_{k=0/1}(\varepsilon_1, \varepsilon_2, a; \Lambda) = \sum_{\ell \in \mathbb{Z} / \mathbb{Z}+\frac{1}{2}} Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, a + \varepsilon_1 \ell; \Lambda e^{t\varepsilon_1/4}) \times Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, a + \varepsilon_2 \ell; \Lambda e^{t\varepsilon_2/4}) \times \frac{\Lambda^{2\ell^2}}{\ell(\varepsilon_1, \varepsilon_2, a)}$$

$\leftarrow$ explicit 1)

Take the limit at $\varepsilon_1=\varepsilon_2=0$:

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\hat{Z}_{k=1}(\varepsilon_1,\varepsilon_2,a,t:\Lambda)}{Z(\varepsilon_1,\varepsilon_2,a:\Lambda)} = \exp(A-B+\frac{t^2}{8}\frac{\partial u}{\partial\log\Lambda})\,\theta_{11}(\frac{\sqrt{-1}}{2\pi}\frac{\partial u}{\partial a}|\tau), \quad \longrightarrow \ast$$

where $\varepsilon_1\varepsilon_2\log Z = F_0 + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2+\varepsilon_2^2}{3}B + \cdots\cdots$

② structure result

Th $\exists!$ class $\Omega_j(\varepsilon,t) \in \mathbb{C}[c_i(\mathcal{E})/[0],\ \varepsilon_1,\varepsilon_2][[t]]$ $(i=2,\cdots,r)$ (indep. of n)

s.t. $$\int_{\hat{M}(2,k,n)} e^{t\mu(C)} = \sum_j \int_{M(2,\hat{n}-j)} \Omega_j(\varepsilon,t)$$

$$\hat{n} = n - \frac{k(r-k)}{2r}$$

$r=2 \Rightarrow c_2(\mathcal{E})/[0]$ is essentially u.

We do not know an explicit formula for $\Omega_j(\varepsilon,t)$ in general, but can compute for lower j, since we can determine it for small n.

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{\hat{Z}_{k=1}}{Z} = -\Lambda t - \frac{t^3}{3!}\Lambda u - \frac{t^5}{5!}\Lambda(u^2+2\Lambda^4) - \frac{t^7}{7!}\Lambda(u^3+6u\Lambda^4) + \cdots\cdot$$

We rewrite $\ast$ in terms of σ-function, and use $$\sigma(t) = t - \frac{g_2}{2}\frac{t^5}{5!} - 6g_3\frac{t^7}{7!} + \cdots$$

$\Rightarrow$ g_2, g_3 are polynomials in u, Λ given above.